

Numerical Solutions of Kähler-Einstein metrics on \mathbb{P}^2 with conical singularities along a conic curve

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Abstract

We solve for the $SO(3)$ -invariant Kähler-Einstein metric on \mathbb{P}^2 with cone singularities along a smooth conic curve using numerical approach. The numerical results show the sharp range of angles $(\pi/4, 2\pi]$ for the solvability of equations, and the right limit metric space $(\mathbb{P}(1, 1, 4))$. These result exactly match our theoretical conclusion.

1 Introduction

Let D be a smooth conic curve in \mathbb{P}^2 . In this work, we fix $D = \{Z_1^2 + Z_2^2 + Z_3^2 = 0\}$. In the recent work [4], we have considered the problem of existence of Kähler-Einstein metrics on \mathbb{P}^2 with cone singularities along D of cone angle $2\pi\beta \in (0, 2\pi]$. The following is the main result in this study [4]:

Theorem 1.1 ([4]). *There exists a conical Kähler-Einstein metric on $(\mathbb{P}^2, (1 - \beta)D)$ if and only if $\beta \in (1/4, 1]$.*

As pointed out to us by Dr. H-J. Hein, when $\beta = \frac{1}{3}$, this gives rise to cone Calabi-Yau cone metric on the 3-dimensional A_2 singularity $x_1^2 + x_2^2 + x_3^2 + x_4^3 = 0$.

This is a question raised by Gauntlett-Martelli-Sparks-Yau in [3]. In [3], they proved there can not exist such Calabi-Yau cone metric on 3-dimensional A_{k-1} singularities $x_1^2 + x_2^2 + x_3^2 + x_4^k = 0$ if $k \geq 4$. The idea is to look at the links L_k of such singularities. Any such Calabi-Yau cone metric would induce a Sasaki-Einstein structure on L_k . By further taking quotient by the $U(1)$ action generated by the natural Reeb vector field, we would get an orbifold Kähler-Einstein metric on $(\mathbb{P}^2, (1 - \frac{1}{k})D)$. In [3], the obstruction for $k \geq 4$ comes from the Lichnerowicz obstruction. In [4] this was explained as $(\mathbb{P}^2, (1 - \frac{1}{k})D)$ being not log-K-stable if $k \geq 4$. For $k = 1$ and $k = 2$ case, we have the standard examples corresponding to the \mathbb{P}^2 with Fubini-Study metric and $(\mathbb{P}^2, \frac{1}{2}D) \cong \mathbb{P}^1 \times \mathbb{P}^1$ with the product metric. These discussion leaves open the existence problem when $k = 3$.

The new insight from [4] is that we can put such kind of orbifold Kähler metrics in the more broad family of conical Kähler metrics. In our notation $\beta = 1/k$. This allows us to give a uniform theory which together with an interpolation argument lead us to Theorem 1.1.

However, as pointed out in [4], such result is in contradiction to the result by Conti in [1], which says there is no cone Calabi-Yau cone metric on A_2 singularities. His proof is by classifying all the cohomogeneity one 5-dimensional Sasaki-Einstein manifolds. This leaves us wondering which one is right.

We decide to attack this question by returning to the approach in [3] where the equations of orbifold Kähler-Einstein metrics on $(\mathbb{P}^2, (1 - 1/k)D)$ were written down. Note that because of $SO(3)$ symmetry, such equation comes from the work in [2]. Moreover, the transformation and change of variables introduced in [3] is very useful for dealing with the problem at hand. In this way, we get a 2nd order differential equation with appropriate boundary conditions.

Since we could not integrate the equation for general β we will use numerical simulation to solve it. This was suggested in [3]. Our goal is to carry out such numerical approach. As it turns out, the result is same as we expected.

Theorem 1.2. *The equations corresponding to $SO(3)$ -invariant Kähler-Einstein metric ω_β on $(\mathbb{P}^2, (1 - \beta)D)$ has a numerical solution if and only if $\beta > 1/4$.*

As suggested by Dr. Song Sun, we will further verify the conjecture proposed in [4] which predicts the limit metric space as β goes to the critical value $1/4$. Again, the numerical result fits well with our expectation.

Theorem 1.3. *As $\beta \rightarrow 1/4$, the metric space $(\mathbb{P}^2, \omega_\beta)$ converges to the metric space $(\mathbb{P}(1, 1, 4), \hat{\omega}_{KE})$ where $\hat{\omega}_{KE}$ is the induced orbifold Kähler-Einstein metric coming from the standard Fubini-Study metric on \mathbb{P}^2 by the natural branch cover: $\mathbb{P}(1, 1, 1) \rightarrow \mathbb{P}^2(1, 1, 4)$.*

The precise meaning of the above statement is detailed in Section 4 and Section 5. This gives direct support to our result in Theorem 1.1. We hope to discuss some generalizations of this example in [4] later.

The organization of this note is as follows. The first section gives a detailed review of the structure of $SO(3)$ -orbits for \mathbb{P}^2 . The second section discusses the equations we want to solve. Again, we carefully review the approach in [3] and work out more details. In the third sections, we show our first numerical result Theorem 1.2. In the last section, after describing the $SU(2)$ -orbits of $\mathbb{P}(1, 1, 4)$ we demonstrate our numerical studies which explains Theorem 1.3.

2 $SO(3)$ orbits

Let us first review how to decompose \mathbb{P}^2 into $SO(3, \mathbb{R})$ -orbits following [3]. First note that $\mathbb{P}^2 = (\mathbb{C}^3 - \{0\})/\mathbb{C}^*$ under the equivalence relation $(Z_1, Z_2, Z_3) \sim (\lambda Z_1, \lambda Z_2, \lambda Z_3)$ for some $\lambda \neq 0 \in \mathbb{C}^*$. Now fix any $0 \neq Z := (Z_i)_{i=1}^3 \in \mathbb{C}^3$, it determines a point in \mathbb{P}^2 with homogeneous coordinate $[Z] := [Z_i]_{i=1}^3 = [Z_1, Z_2, Z_3]$. Now write the polar decomposition

$$Z_1^2 + Z_2^2 + Z_3^2 = \rho^2 e^{2i\theta}.$$

So if we define

$$\tilde{Z}_i = e^{-i\theta} Z_i,$$

then $[Z_i]_{i=1}^3 = [\tilde{Z}_i]_{i=1}^3$ and

$$\tilde{Z}_1^2 + \tilde{Z}_2^2 + \tilde{Z}_3^2 = \rho^2 \geq 0. \quad (1)$$

Now write

$$\tilde{Z}_i = u_i + \sqrt{-1}v_i,$$

then the identity (1) is equivalent to the identity

$$|u|^2 - |v|^2 = r^2; \quad u \cdot v = 0. \quad (2)$$

We use these two relations to define the set:

$$\mathbb{O} = \{(u, v) \neq 0 \mid u \cdot v = 0, |u|^2 - |v|^2 \geq 0\} \subset (\mathbb{R}^3)^2 - \{0\}.$$

Define an equivalence relation on \mathbb{O} by $(u, v) \sim a(u, v)$ for some $a \in \mathbb{R}^*$. Denote the quotient set by $\overline{\mathbb{O}} = \mathbb{O} / \sim$. Then we have defined a homeomorphism

$$\begin{aligned} \Phi : \mathbb{P}^2 &\longrightarrow \overline{\mathbb{O}} \\ [Z_i]_{i=1}^3 &\mapsto [u, v] \text{ satisfying } u + \sqrt{-1}v = e^{-\frac{i}{2}\text{Arg}(Z_1^2 + Z_2^2 + Z_3^2)}(Z_1, Z_2, Z_3). \end{aligned}$$

The $SO(3)$ acts on $\mathbb{P}^2 \cong \overline{\mathbb{O}}$ by

$$g \cdot (u, v) = (gu, gv).$$

The quotient of this action is an interval:

$$\begin{aligned} R : \overline{\mathbb{O}} &\longrightarrow [0, 1] \\ [u, v] &\mapsto \frac{|v|}{|u|} \end{aligned}$$

So the function R classifies $SO(3)$ orbit. Moreover it's easy to verify that equivalently we have the relation

$$\frac{|Z_1^2 + Z_2^2 + Z_3^2|}{|Z_1|^2 + |Z_2|^2 + |Z_3|^2} = \frac{1 - R^2}{1 + R^2}. \quad (3)$$

Now at each point $(v, w) \in \mathbb{O}$, we can get a orthonormal basis $(e_u = u/|u|, e_v = v/|v|, e_w := e_u \times e_v)$. We will denote $U(1)_1, U(1)_2$ and $U(1)_3$ to be the rotation around the axes in the direction e_u, e_v and e_w respectively.

Lemma 2.1. *The generic orbit is $\text{Orb}_{R=R_0} = SO(3)/\mathbb{Z}_2$ (when $0 < R_0 = R([u, v]) < 1$). The two special orbits are*

$$\begin{aligned} \text{Orb}_{R=0} &= (SO(3)/\mathbb{Z}_2)/U(1)_1 = \mathbb{RP}^2 \\ \text{Orb}_{R=1} &= (SO(3)/\mathbb{Z}_2)/U(1)_3 = \mathbb{P}^1 \end{aligned}$$

Proof. When $0 < R = \frac{|v|}{|u|} < 1$, the stabilizer of $SO(3)$ action at $[v, w]$ is isomorphic to \mathbb{Z}_2 with generator being the rotation around e_w with angle π , i.e. $(e_u, e_v, e_w) \rightarrow (-e_u, -e_v, e_w)$.

When $R = 0, v=0$. The stabilizer is generated by \mathbb{Z}_2 and $U(1)_1$. The generator of \mathbb{Z}_2 can be chosen to be $(e_u, e_2, e_3) \mapsto (-e_u, -e_2, e_3)$ for any e_2, e_3 such that $\{e_u, e_2, e_3\}$ is an orthonormal basis. $U(1)_1$ is the rotation group around e_u . It's easy to verify that

$$\text{Orb}_{R=0} = (\mathbb{R}^3 - \{0\})/\mathbb{R}^* = \mathbb{RP}^2.$$

When $R = 1, |u| = |v|$. The stabilizer is $U(1)$ -rotation group around e_w denoted as $U(1)_3$. Note $\mathbb{Z}_2 \subset U(1)_3$. It's easy to see that (for example by (3))

$$\text{Orb}_{R=1} = \{Z_1^2 + Z_2^2 + Z_3^2 = 0\} \cong \mathbb{P}^1 \subset \mathbb{P}^2.$$

□

Fix the generator of $\mathfrak{so}(3) = \text{Lie}(SO(3))$ to be

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the corresponding invariant vector field on the orbit $SO(3)([u, v])$ at point $[u, v]$ is given by the infinitesimal rotation around three axes in the directions of e_u, e_v, e_w respectively. In other words, they are the generator of the action of $U(1)_1, U(1)_2, U(1)_3$ respectively.

1. Around e_u :

$$T_u = \left. \frac{d}{d\theta} \right|_{\theta=0} (u + \sqrt{-1}(\cos \theta e_v + \sin \theta e_w)|v|) = \sqrt{-1}|v|e_w.$$

2. Around e_v : $T_v = \left. \frac{d}{d\theta} \right|_{\theta=0} (-\sin \theta e_w + \cos \theta e_u)|u| + \sqrt{-1}v = -|u|e_w.$

3. Around e_w :

$$\begin{aligned} T_w &= \left. \frac{d}{d\theta} \right|_{\theta=0} (|u|(\cos \theta e_u + \sin \theta e_v) + \sqrt{-1}(-\sin \theta e_u + \cos \theta e_v)|v|) \\ &= |u|e_v - \sqrt{-1}|v|e_u. \end{aligned}$$

We can define another vector field generating the radial transformation

$$T_R = \frac{d}{d\theta} \Big|_{\theta=0} \left(|u|(e_u + \sqrt{-1} \left(\frac{|v|}{|u|} + \theta \right) e_v) \right) = \sqrt{-1}|u|e_v.$$

Note that strictly speaking, the above vectors represent the tangent vector in

$$T_{[u+iv]\mathbb{P}^2} = \text{Hom}(\mathbb{C}(u, v), \mathbb{C}^{n+1}/\mathbb{C}(u, v)).$$

Lemma 2.2. *On $\text{Orb}_{R=0} = \mathbb{RP}^2$, $T_u = 0$; On $\text{Orb}_{R=1} = \mathbb{P}^1$, $T_w = 0$.*

Proof. When $R = 0$, $|v| = 0$, so $T_u = 0$ on $\text{Orb}_{R=0}RP^2$. When $R = 1$,

$$T_w = |u| \frac{v}{|v|} - \sqrt{-1}|v| \frac{u}{|u|} = -\sqrt{-1}(u + \sqrt{-1}v)$$

so $T_w = -\sqrt{-1}(u + \sqrt{-1}v) \in \mathbb{C} \cdot (u, v)$, so $T_w|_{R=1} = 0$, i.e. T_w vanishes on the special orbits $\text{Orb}_{R=1} = \mathbb{P}^1$. \square

Note that this Lemma also follows from Lemma 2.1 by the fact that $U(1)_1$ is the stabilizer group on $\text{Orb}_{R=0}$ generated by T_u , while $U(1)_3$ is the stabilizer group on $\text{Orb}_{R=1}$ generated by T_w .

3 Equations for $SO(3)$ invariant Kähler-Einstein

For special metrics g on \mathbb{P}^2 , we have the following

Lemma 3.1. 1. *For any Kähler metric g , we have $|T_u|_g \leq |T_v|_g$. The equality holds only on the special orbit $\text{Orb}_{R=1} = \mathbb{P}^1$.*

2. *For any $SO(3)$ invariant metric g , $|T_v|_g = |T_w|_g$ on the special orbit $\text{Orb}_{R=0} = \mathbb{RP}^2$.*

Proof. 1. Because Kähler metric is compatible with complex structure $J = i \cdot$, so

$$0 \leq \frac{|T_u|_g}{|T_v|_g} = \frac{|i|v|e_w|_g}{||u|e_w|_g} = \frac{||v|e_w|_g}{||u|e_w|_g} = \frac{|v|}{|u|} = R \leq 1. \quad (4)$$

2. On the special orbit $\text{Orb}_{R=0} = \mathbb{RP}^2$, $v = 0$. Let $\gamma_1(\theta) = |u|(-\sin \theta e_w + \cos \theta e_u)$ and $\gamma_2(\theta) = |u|(\cos \theta e_u + \sin \theta e_w)$. Then $T_v = \gamma'_1(0)$ and $T_w = \gamma'_2(0)$. Because there exist rotations $g(\theta)$ in $SO(3)$ such that $g(\theta) \cdot \gamma_1(\theta) = \gamma_2(\theta)$. So the conclusion follows from invariance of the metric under $SO(3)$. \square

Now choose the dual basis of $\{T_R, T_u, T_v, T_w\}$ to be one forms given by $\{dR, \sigma_1, \sigma_2, \sigma_3\}$. For any $SO(3)$ invariant Kähler metric on \mathbb{P}^2 , $\{T_u, T_v, T_w, T_R\}$ is orthogonal. The metric can be written in the form

$$g = (dt)^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2. \quad (5)$$

where

$$dt = -|T_R|_g dR, \quad a = |T_u|_g, \quad b = |T_v|_g, \quad c = |T_w|_g.$$

The minus sign in the first identity is to make the special orbit \mathbb{P}^1 to sit in the distance 0 location. By Lemma 2.2 and Lemma 3.1, we know that

Corollary 3.1. 1. *$a \leq b$ on \mathbb{P}^2 . On $\text{Orb}_{R=1} = \mathbb{P}^1$, $c = 0$, $a = b$.*

2. *On $\text{Orb}_{R=0} = \mathbb{RP}^2$, $a = 0$, $b = c$.*

Example 3.1. When $\beta = 1$, then the $SO(3)$ invariant metric is the standard Fubini-Study metric on \mathbb{P}^2 . We can write it in the form of (5). One way to do this is to recall the following description of Study-Fubini metric. Let $\gamma(t) := [Z_1(t), Z_2(t), Z_3(t)]$ be a curve in \mathbb{P}^2 with the tangent vector is $\gamma'(0) = ((Z_1(0), Z_2(0), Z_3(0)) \mapsto (Z'_1(0), Z'_2(0), Z'_3(0))) \in \text{Hom}(\mathcal{O}_{\mathbb{P}^2}(1), \mathbb{C}^{n+1}/\mathcal{O}_{\mathbb{P}^2}(1))$. The length of $\gamma'(0)$ is given by

$$|\gamma'(0)|_{FS}^2 = \frac{|Z'(0)^\perp|^2}{|Z(0)|^2} = \left(|Z'(0)|^2 - \frac{|\langle Z'(0), Z(0) \rangle|^2}{|Z(0)|^2} \right) / |Z(0)|^2.$$

Using this formula, it's easy to verify that

$$\begin{aligned} |T_R|_{FS} &= \frac{|u|^2}{|u|^2 + |v|^2} = \frac{1}{1+R^2}, & |T_u|_{FS} &= \frac{|v|}{\sqrt{|u|^2 + |v|^2}} = \frac{R}{\sqrt{1+R^2}} \\ |T_v|_{FS} &= \frac{|u|^2}{|u|^2 + |v|^2} = \frac{1}{\sqrt{1+R^2}}, & |T_w|_{FS} &= \frac{|u|^2 - |v|^2}{|u|^2 + |v|^2} = \frac{1-R^2}{1+R^2}. \end{aligned}$$

So the normal distance function t is determined by

$$dt = -\frac{1}{1+R^2}dR \quad \& \quad t(1) = 0 \implies R = \tan\left(\frac{\pi}{4} - t\right).$$

So $0 \leq t \leq \pi/4$ and

$$a = \sin\left(\frac{\pi}{4} - t\right) = \cos\left(t + \frac{\pi}{4}\right), \quad b = \sin\left(t + \frac{\pi}{4}\right), \quad c = \cos\left(\frac{\pi}{2} - 2t\right) = \sin(2t).$$

Example 3.2. Without further ado, we give the data corresponding to $\mathbb{P}^1 \times \mathbb{P}^1 = (P^2, \frac{1}{2}D)$. (See [2] and [3])

$$a(t) = \frac{1}{\sqrt{3}} \cos(\sqrt{3}t), \quad b(t) = \frac{1}{\sqrt{3}}, \quad c(t) = \frac{1}{\sqrt{3}} \sin(\sqrt{3}t).$$

The range for t is $0 \leq t \leq \pi/(2\sqrt{3})$.

By [2] and [3], the Kähler-Einstein equation is reduced to a system of ODEs:

$$\begin{cases} \dot{a} &= -\frac{b^2+c^2-a^2}{2bc} \\ \dot{b} &= -\frac{a^2+c^2-b^2}{2ac} \\ \dot{c} &= -\frac{a^2+b^2-c^2}{2ab} + 6ab \end{cases} \quad (6)$$

Note that the equation in [3] differs from [2] by a (negative) factor $(-abc)$ which is caused by a change of variable.

The boundary condition is given at $t = 0$ corresponds to the special orbit $\text{Orb}_{R=1} = \mathbb{P}^2$ where by Corollary 3.1 $a = |T_u|_g = |T_v|_g = b$ and $c = |T_w|_g = 0$. Moreover, the cone angle equal to $2\pi\beta$ along $\text{Orb}_{t=0} = \mathbb{P}^1$ requires $\dot{c} = 2\beta$. The factor 2 comes from the fact that when $0 < R < 1$ the stabilizer is \mathbb{Z}_2 . So the boundary is given

$$\begin{aligned} a(t) &= \alpha + O(t) \\ b(t) &= \alpha + O(t) \\ c(t) &= 2\beta t + O(t^2) \end{aligned}$$

Since the normalized Kähler-Einstein metric ω'_β satisfies

$$\text{Ric}(\omega'_\beta) = \omega'_\beta + (1 - \beta)\{D\}.$$

Because $[D] = \frac{2}{3}c_1(\mathbb{P}^2)$, so, by taking cohomological classes on both sides, we get

$$[\omega'_\beta] = \frac{1}{3}(1 + 2\beta) \cdot c_1(\mathbb{P}^2).$$

So α and β are related by $\alpha^2 = \delta \cdot \frac{1}{3}(1 + 2\beta)$ since both sides are proportional to the volume of \mathbb{P}^1 . The factor δ can be carefully tracked out, but it can also be easily determined either by checking the standard \mathbb{P}^2 with Fubini-Study metric in Example 3.1 or by substituting in to the last equation in (6). The result is

$$\alpha^2 = \frac{1}{6}(1 + 2\beta).$$

obtained from equation (6).

When $t = t_*$, we know from Corollary 3.1 that $a(t_*) = 0$ and $b(t_*) = c(t_*)$.

Lemma 3.2.

$$\dot{a}(t_*) = -1, \quad \dot{b}(t_*) = \dot{c}(t_*) = 0. \quad (7)$$

Proof. From the first equation in (6) and $b(t_*) = c(t_*)$, we get $\dot{a}(t_*) = -1$. Then we use this to derive from Equation (6) that

$$\dot{b}(t_*) = -\dot{c}(t_*) = \lim_{t \rightarrow t_*} \frac{b - c}{a} = -(\dot{b}(t_*) - \dot{c}(t_*)) = -2\dot{b}(t_*).$$

So the 2nd identity follows. \square

Note that $\dot{a}(t_*) = -1$ is compatible with the fact that the metric is smooth along $\text{Orb}_{R=0} \cong \mathbb{RP}^2$.

Note the solutions of equation (6) is not unique around the point $(a(0), b(0), c(0)) = (\alpha, \alpha, 0)$. There are at least three possibilities: $a \leq b$, $a = b$, $a \geq b$. The $a = b$ case corresponds to the Gibbons-Pope-Pederson metric as pointed out in [2]. We are in the $a \leq b$ case. The symmetry of a, b is broken by writing down the differential equation for the variable $R = a/b$. Using (6), we get

$$c \frac{d}{dt} \left(\frac{a}{b} \right) = \left(\frac{a}{b} \right)^2 - 1.$$

So it's natural to do the following change of variables introduced by [3].

$$\frac{dr}{dt} = 1/c.$$

Then

$$\frac{dR}{dr} = R^2 - 1.$$

Using $a \leq b$ ((4)), we get the solution

$$R = \frac{a}{b} = -\tanh(r).$$

Moreover, we get the range for r : $-\infty < r \leq 0$. We list the the ranges of R, t, r as follows:

	\mathbb{P}^2	$SO(3)/\mathbb{Z}_2$	RP^2
R	$R = 1$	$1 > R > 0$	$R = 0$
t	$t = 0$	$0 < t < t_*$	$t = t_*$
r	$r = -\infty$	$-\infty < r < 0$	$r = 0$

Define $f = ab$, then f satisfies the second order differential equation

$$\frac{d}{dt} \log \left(f \frac{df}{dr} \right) = 2[6f + \coth(2r)].$$

Example 3.3. By easy calculations, one can get that, for \mathbb{P}^2 , $f = -\frac{1}{2} \tanh(2r)$, $f_r(0) = 1$; and for $\mathbb{P}^1 \times \mathbb{P}^1$, $f = -\frac{1}{3} \tanh(r)$, $f_r(0) = \frac{1}{3}$. See [3].

Let $h = f_r$ then this is equivalent to a system:

$$\begin{cases} f_r &= h \\ h_r &= 12fh + 2 \coth(2r)h - \frac{h^2}{f}. \end{cases} \quad (8)$$

It's easy to verify that the data (f, R, h) and (a, b, c) determine each other by the relation

$$f = ab, \quad R = \frac{a}{b}, \quad h = f_r = -c^2. \quad (9)$$

The boundary condition is given by

$$f(-\infty) = \alpha^2, f(0) = 0.$$

$$h(0) = f_r(0) = -c(t_*)^2 = -b(t_*)^2.$$

Using (7), (6) and $t_r(t) = c(t)$, we get

$$h_r(0) = f_{rr}(0) = (f_{tt}t_r + f_t(t_r)_t)t_r|_{t=t_*} = \ddot{a}(t_*)b(t_*)^3 = 0.$$

4 Numerical Studies: $\beta > 1/4$

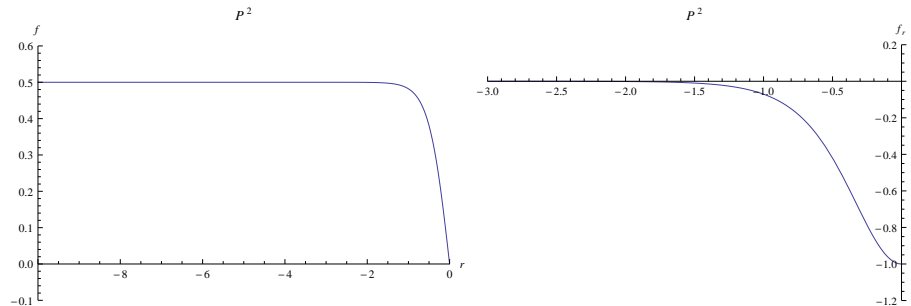
Now we explain our numerical simulation. We introduce the variable τ for convenience and choose boundary value $(f(0), h(0)) = (0, -\frac{1}{\tau} := -b(t_*)^2)$ and solve the equation (8) numerically. However, this can not be done because there is a zero on the denominator for $r = 0$ on the second equation in (8) (although it's canceled by zero on the numerator). We can however move away from $r = 0$ a little bit by using the boundary condition and Taylor expansion:

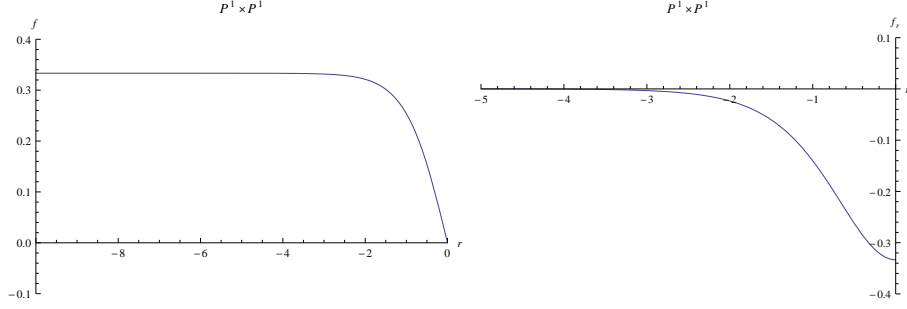
$$\begin{aligned} f(r) &= f(0) + f_r(0)r + O(r^2) = -\frac{1}{\tau}r + O(r^2) \\ h(r) &= h(0) + h_r(0)r + O(r^2) = -\frac{1}{\tau} + O(r^2) \end{aligned}$$

So numerically, we can choose $r_0 < 0$ to be very close to 0 and choose the boundary condition to be

$$(f(r_0), h(r_0)) = (-\frac{r_0}{\tau}, -\frac{1}{\tau}).$$

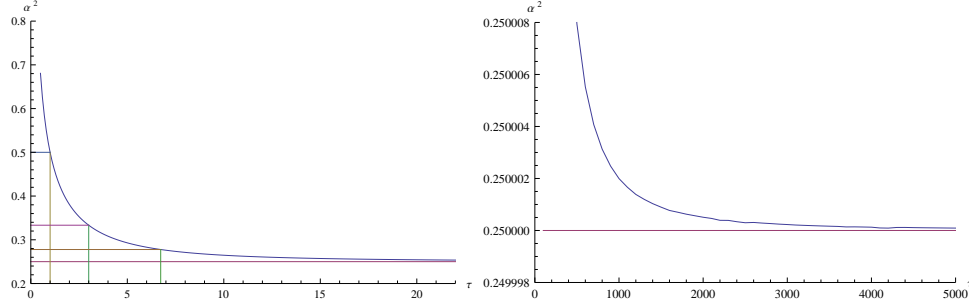
For example, in the following numerical simulation, we choose $r_0 = -10^{-5}$. Then we can shoot the trajectory out for r going from r_0 backward to $-\infty$. The following numerical solutions correspond to \mathbb{P}^2 when $\tau = 1$ and $\mathbb{P}^1 \times \mathbb{P}^1 = (P^2, \frac{1}{2}D)$ when $\tau = 3$ respectively. They can be obtained for example by the **NDSolve** tool in **Mathematica**.





Of course, the above graphs of $f = f(r)$ just recover the graph $f(r) = -\frac{1}{2} \tanh(2r)$ for \mathbb{P}^2 and $f(r) = -\frac{1}{3} \tanh(r)$ for $\mathbb{P}^1 \times \mathbb{P}^1$ (up to high precision).

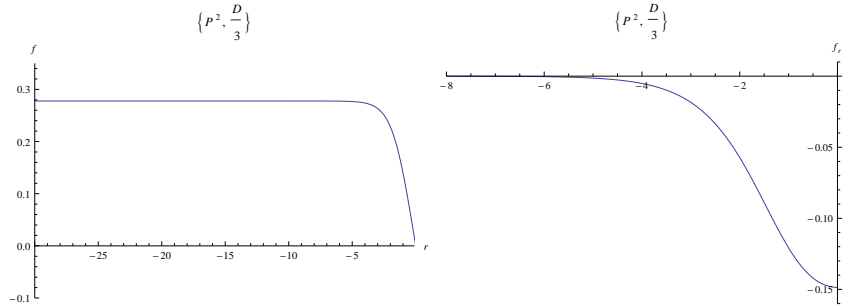
If we choose different τ , then we get different solution $f, h = f_r$. We know that $\lim_{r \rightarrow -\infty} f(r) = f(-\infty) = \alpha^2 = \frac{1+2\beta}{6}$. Numerically, we can just evaluate $f(r)$ for r being sufficiently negative to calculate α^2 . Actually, after several tests, one can observe that for fixed τ , the graph becomes flat very quickly which means $f(r)$ becomes stabilized very quickly as r goes toward $-\infty$. We can use **Mathematica** to calculate (very dense) sequences of data for $\{\tau, f(\tau, r)\}$ where we make solution f depend the boundary data τ . Then we sample the value of $f(\tau, r)$ at $r = -300$. (One can certainly choose r to be more negative but the visual effect does not change) The following is the numerical result. The two pictures are for short range and long range of τ respectively.



We see immediately that α^2 is a decreasing function of τ . More importantly, from the picture, we see that one always has

$$\alpha^2 = \frac{1+2\beta}{6} > 0.25 \iff \beta > \frac{1}{4}.$$

and all the $\beta > \frac{1}{4}$ can be achieved. In particular, when $\beta = \frac{1}{3}$, where $\alpha^2 = \frac{5}{18} = 0.277777\dots$, one can find approximate value of $\tau \sim 0.673$ from numerical result. In the picture, we have identified three special points: $(1, 0.5), (3, 1/3)$ and $(0.673, \frac{5}{18})$ which corresponds to $\beta = 1, \frac{1}{2}$ and $\frac{1}{3}$ respectively. The corresponding graph of f and $g = f_r$ for $\tau = 6.73(\beta = \frac{1}{3})$ is the following



Finally, note that we are only interested when $\beta \leq 1$, or equivalently when $\alpha^2 \leq 0.5$. However the picture suggests we can even pass $\beta \leq 1$ and solve for conic Kähler-Einstein metric with cone angle $2\pi\beta > 2\pi$ along the conic curve.

5 Limit as β goes to $1/4$

We know that $SU(2)$ acts on \mathbb{P}^1 naturally. As pointed out in [4], the following embedding is equivariant with respect to the covering homomorphism $\phi : SU(2) \rightarrow SO(3, \mathbb{R})$.

$$\begin{aligned} \Delta : \mathbb{P}^1 &\longrightarrow \mathbb{P}^2 \\ [U_0, U_1] &\mapsto [U_0^2 + U_1^2, 2iU_0U_1, i(U_0^2 - U_1^2)]. \end{aligned}$$

Here $SU(2)$ acts on $\mathbb{P}^2(1, 1, 4)$ by acting on the first two variables:

$$g \cdot [U_0, U_1, V] = [g \cdot (U_0, U_1), V].$$

Note that

$$\Delta(\mathbb{P}^1) = \{Z_1^2 + Z_2^2 + Z_3^2 = 0\}.$$

Fix generators of $SU(2, \mathbb{C})$ to be standard Pauli matrices:

$$Y_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Y_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, Y_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Note that the commutator relation $[Y_1, Y_2] = 2Y_3$ and cyclicly. So by letting $\tilde{Y}_i = \frac{Y_i}{2}$, \tilde{Y}_i 's satisfy $[\tilde{Y}_1, \tilde{Y}_2] = \tilde{Y}_3$ and cyclicly. Let \tilde{Y}_i^* be the vector fields on $\mathbb{P}(1, 1, 4)$ corresponding to the infinitesimal actions of \tilde{Y}_i . Then we have

Lemma 5.1. *When we restrict to \mathbb{P}^1 , $\Delta_*\tilde{Y}_1^* = T_u$, $\Delta_*\tilde{Y}_2^* = -T_v$, $\Delta_*\tilde{Y}_3^* = -T_w$.*

Proof. $\Delta(1, 0) = (1, 0, i) = u + iv$ with $u = (1, 0, 0)$ and $v = (0, 0, 1)$. So $w = u \times v = -(0, 1, 0)$.

$$\Delta_*\tilde{Y}_i^* = (2(U_0\dot{U}_0 + U_1\dot{U}_1), 2i(\dot{U}_0U_1 + U_0\dot{U}_1), 2i(U_0\dot{U}_0 - U_1\dot{U}_1))$$

1. $\tilde{Y}_1^* = \frac{1}{2}(U_1, -U_0)$, so

$$\Delta_*\tilde{Y}_1^* = (0, -i(U_0^2 - U_1^2), 2iU_0U_1).$$

In particular, $\tilde{Y}_1^*|_{(1,0)} = \frac{1}{2}(0, -1)$ and $\Delta_*\tilde{Y}_1^*|_{(1,0)} = (0, -i, 0) = ie_w$. So $\Delta_*\tilde{Y}_1^* = T_u$.

2. $\tilde{Y}_2^* = \frac{1}{2}(iU_1, iU_0)$, so

$$\Delta_*\tilde{Y}_2^* = (2iU_0U_1, -(U_0^2 + U_1^2), 0).$$

In particular, $\tilde{Y}_2^*|_{(1,0)} = \frac{1}{2}(i, 0)$, $\Delta_*\tilde{Y}_2^*|_{(1,0,i)} = (0, -1, 0) = e_w$. So $\Delta_*\tilde{Y}_2^* = -T_v$.

3. $\tilde{Y}_3^* = \frac{1}{2}(iU_0, -iU_1)$, so

$$\Delta_*\tilde{Y}_3^* = (i(U_0^2 - U_1^2), 0, -(U_0^2 + U_1^2)).$$

In particular, $\tilde{Y}_3^*|_{(1,0)} = \frac{i}{2}(1, 0)$ and $\Delta_*\tilde{Y}_3^*|_{(1,0,i)} = (i, 0, -1) = -v + iu$. So $\Delta_*\tilde{Y}_3^* = -T_w$.

□

We can define a function which classifies the $SU(2)$ -orbits

$$\begin{aligned} \tilde{R} : \mathbb{P}(1, 1, 4) &\longrightarrow [0, +\infty) \\ [U_0, U_1, V] &\mapsto \left(\frac{|U_0|^2 + |U_1|^2}{|V|^{1/2}} \right)^{1/2} \end{aligned}$$

Lemma 5.2. *The generic orbit when $0 < R < \infty$ is isomorphic to $SU(2)/\mathbb{Z}_4 \cong SO(3)/\mathbb{Z}_2$. The special orbit are*

$$\text{Orb}_{\tilde{R}=0} = \text{Pt} = [0, 0, 1], \quad \text{Orb}_{\tilde{R}=\infty} = \mathbb{P}^1.$$

Proof. If $0 < R < +\infty$, then $[U_0, U_1, V]$ is the same as $[\sqrt{-1}^j U_0, \sqrt{-1}^j U_1, V]$, $j = 1, 2, 3, 4$. So the stabilizer is isomorphic to \mathbb{Z}_4 . The cases of special orbits are clear. \square

Now the $SU(2)$ -invariant Kähler metric has the form

$$g = dt^2 + a^2 \sigma_1^2 + b^2 \sigma_2^2 + c^2 \sigma_3^2.$$

Similar as the example 3.1 in Section 2, we can calculate the induced orbifold Kähler-Einstein metric by the branch covering map:

$$\begin{aligned} \mathbb{P}(1, 1, 1) &\longrightarrow \mathbb{P}(1, 1, 4) \\ [Z_1, Z_2, Z_3] &\mapsto [Z_1, Z_2, Z_3^4]. \end{aligned}$$

Because the metric is $SU(2)$ invariant, to write down the metric we only need to calculate the length of the basic vector fields at the the special point $(\tilde{R}, 0, 1)$ in each $SU(2)$ -orbit.

1. $T_{\tilde{R}}|_{(\tilde{R}, 0, 1)} = (1, 0, 0)$, $|T_{\tilde{R}}| = \frac{1}{1+\tilde{R}^2}$.
2. $\tilde{Y}_1^*|_{(\tilde{R}, 0, 1)} = \frac{1}{2}(0, -\tilde{R}, 0)$, $a = |\tilde{Y}_1^*|_g = \frac{1}{2} \frac{\tilde{R}}{\sqrt{1+\tilde{R}^2}}$.
3. $\tilde{Y}_2^*|_{(\tilde{R}, 0, 1)} = \frac{1}{2}(0, i\tilde{R}, 0)$, $b = |\tilde{Y}_2^*|_g = \frac{1}{2} \frac{\tilde{R}}{\sqrt{1+\tilde{R}^2}}$.
4. $\tilde{Y}_3^*|_{(\tilde{R}, 0, 1)} = \frac{1}{2}(i\tilde{R}, 0, 0)$, $c = |\tilde{Y}_3^*|_g = \frac{1}{2} \frac{\tilde{R}}{1+\tilde{R}^2}$.

Again, we can transform to the distance function:

$$dt = -\frac{1}{1+\tilde{R}^2} \& \tilde{R}(+\infty) = 0 \implies \tilde{R} = \tan(\pi/2 - t), 0 \leq t \leq \pi/2.$$

By substituting \tilde{R} into the expression of a , b and c , we get the data for $\mathbb{P}(1, 1, 4)$:

$$\begin{aligned} a = b &= \frac{1}{2} \sin\left(\frac{\pi}{2} - t\right) = \frac{1}{2} \cos(t). \\ c &= \frac{1}{4} \sin(\pi - 2t) = \frac{1}{4} \sin(2t). \end{aligned}$$

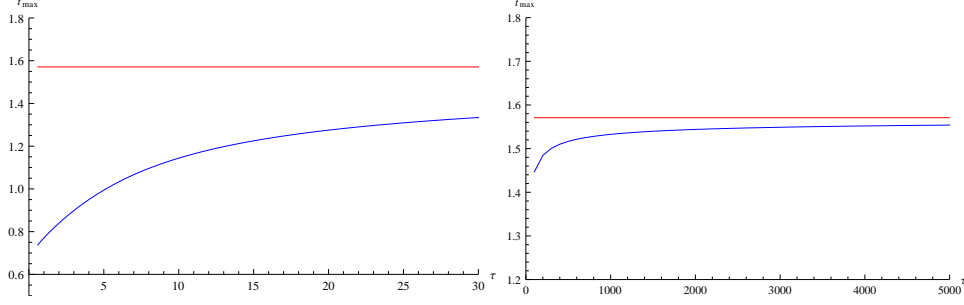
Note that in this case, $a/b \equiv 1$. This is very different from the case where $\beta > 1/4$. For the latter, $a < b$ except on the special fibre $\text{Orb}_{R=1} \cong \mathbb{P}^1$ where $a = b$. Moreover, the boundary condition now becomes

$$\begin{aligned} a(t) = b(t) &= 1/2 + O(t^2) \\ c(t) &= \frac{1}{2}t + O(t^3) \end{aligned}$$

On the other end where $t_* = \pi/2$, $a(\pi/2) = b(\pi/2) = c(\pi/2) = 0$. Geometrically, the special fibre $\text{Orb}_{R=0} \cong \mathbb{RP}^2$ shrinks to a point as $\beta \rightarrow 1/4$. If we do the same transformation that $dr/dt = 1/c$, the range of r will becomes $(-\infty, +\infty)$ instead of $(-\infty, 0)$ because $c(t_*) = 0$.

Next we give the numerical results which show that the metric ω_β converges to the orbifold Kähler-Einstein metric on $\mathbb{P}(1, 1, 4)$.

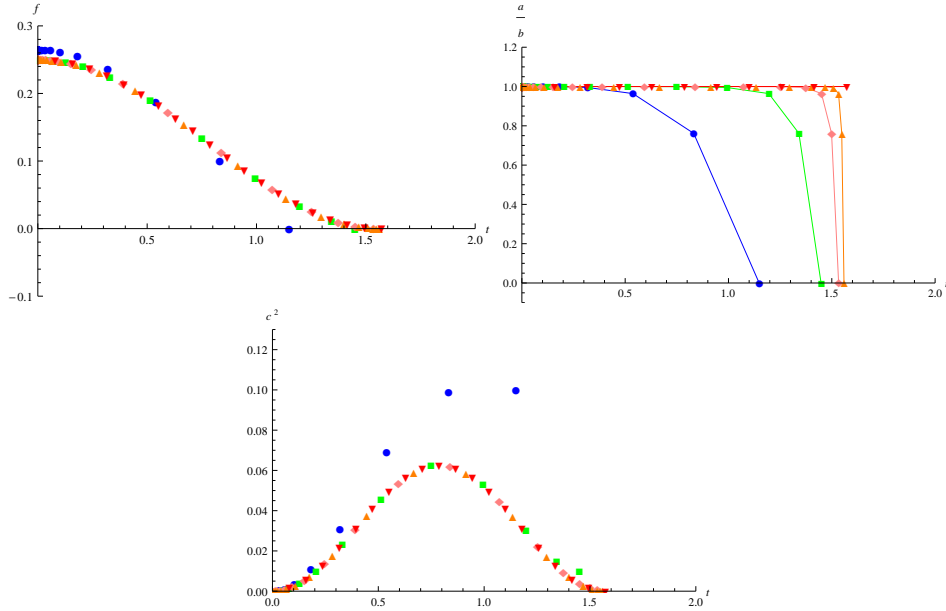
First we integrate the identity $dr/dt = 1/c$ numerically and plot the relation between the boundary value $b(t_*)^2 = 1/\tau$ and $t_{\max} = t_*$. We see that the maximal value for t is an increasing function of τ . As $\tau \rightarrow +\infty$, or equivalently as $\beta \rightarrow 1/4$, $t_{\max} = t_*$ converges to $\pi/2$.



Note that the coordinate t is the distance function from the special orbit \mathbb{P}^1 . So t is a geometrically meaningful coordinate in contrast with r which is only an auxiliary coordinate. So we can get a good convergence only when we look the data as functions t .

Now we can plot the graph of the data set $(f = ab, R = a/b, -f_r = c^2)$ as the function of t instead of r . (See (9)). The following picture shows the data for four τ 's: $\tau = 10^i$ for $i = 1, 2, 3, 4$. The corresponding colors and markers are “Blue Round”, “Green Square”, “Orange Triangle”, “Pink Diamond” for $i = 1, 2, 3, 4$ respectively. The “Red Upside-down Triangle” represent the data for $\mathbb{P}(1, 1, 4)$ where

$$f(t) = a(t)b(t) = \frac{1}{4} \cos^2(t), \quad R = \frac{a}{b} \equiv 1, \quad c^2(t) = \frac{1}{4} \sin^2(2t).$$



One can see that the data for τ large fits with the data for $\mathbb{P}(1, 1, 4)$ very well. Again, we know that τ going to $+\infty$ is equivalent to β going to $1/4$. So the numerical result implies the expected result: as $\beta \rightarrow 1/4$, the metric ω_β converges to the orbifold Kähler-Einstein metric $\hat{\omega}_{KE}$ on $P(1, 1, 4)$.

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